AD-A209 977

CCS Research Report 611

POLYHEDRAL CONE-RATIO DEA MODELS WITH AN ILLUSTRATIVE APPLICATION TO LARGE COMMERCIAL BANKS

by

A. Charnes
W. W. Cooper
D. B. Sun*
Z. M. Huang

CENTER FOR CYBERNETIC STUDIES
The University of Texas
Austin, Texas 78712

DTIC
S ELECTE D
JUL 12 1989
H

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

CCS Research Report 611

POLYHEDRAL CONE-RATIO DEA MODELS WITH AN ILLUSTRATIVE APPLICATION TO LARGE COMMERCIAL BANKS

by

A. Charnes
W. W. Cooper
D. B. Sun*
Z. M. Huang

October 1988

*University of Maine

To be presented at the NSF Conference on Parametric and Nonparametric Approaches to Frontier Analysis on September 30-October l, 1988, at the University of North Carolina at Chapel Hill. An earlier version was presented at the ORSA/TIMS Meeting in St. Louis, Missouri in October, 1987.

NOTE: The research for this paper was partly supported by National Science Foundation Grant SES 8520806 with the Center for Cybernetic Studies at The University of Texas at Austin. It was also partly supported by the IC2 Institute at The University of Texas. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

CENTER FOR CYBERNETIC STUDIES

A. Charnes Director

College of Business Administration, CBA 5.202
The University of Texas at Austin
Austin, Texas 78712-1177
(512) 471-1821

DTIC
S ELECTE D
JUL 12 1989
H

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

ABSTRACT

This study is concerned with developing tools which can be applied to managerial performance evaluation and/or for directing audit activities for profit-making as well as not-for-profit entities using multiple inputs to produce multiple outputs. This is accomplished by building on basic concepts of Data Envelopment Analysis (DEA). In particular, the customary "CCR ratio forms" as described in Charnes, Cooper and Rhodes (1978), are here extended in the polyhedral case specialization of the new "cone-ratio form" of Charnes, Cooper, Wei and Huang (1986). Numerical examples are supplied along with mathematical developments and geometric portrayals of what is involved, and this is followed by an example application to the evaluation of large commercial U.S. banks as drawn from Sun (1987).

KEY WORDS:

Efficiency,

Data Envelopment Analysis,

Multi-attribute Optimization

Polyhedral Cones

Polar Cones.

DTIC COPY INSPECTED 1

Accession For	
NTIS GRA&I	☑
DTIC TAB	☐
Unannounced	☐
Justification	
By per form 50	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

INTRODUCTION:

DEA models as in the CCR ratio form of Charnes, Cooper, Rhodes (l978), sometimes rate many DMUs (Decision Making Units) as efficient when there are reasons to suppose that this rating is not warranted–e.g., as determined from expert opinions. An "overabundance" of such efficiently rated DMUs may be related to the fact that the dual evaluators assigned for the inputs and outputs themselves do not explicitly take some a priori conditions into account. Any DMU not dominated in some input or output could then be rated as efficient. That is, a DMU could be rated as l00% efficient by virtue of being sufficiently strong in only a single input or output, even if that input or output seems relatively unimportant to persons who are experts in the industry in which this DMU is located.

Figure l presents an illustration with output isoquant drawn in the input space. The five DMUs on the isoquant line, A1, . . ., A5, are all technically efficient. However, if the prevalent price ratio is p_l/p_2, as shown, then only Al, A2 (and their convex combinations) are both technically and allocatively (price) efficient. Furthermore, A4 becomes less desirable economically than A6, as shown by the broken lines, even though A6 is not technically efficient. Thus, only a subset of technically efficient DMUs may also be economically "efficient" (or viable) in that they fall in the economically viable range for the marginal contributions of inputs.

A method is desired which can distinguish economically viable DMUs from DMUs which are only technically efficient, and these results can be used to reevaluate other DMUs. This can be provided by suitably restricting the cones of input-output structure used in the multi-criteria optimizations of DEA, which is the key feature of the cone-ratio DEA models --*viz.*, they impose relevant constraints on the optimal conical (or convex) combinations of inputs and outputs through the use of polyhedral cones of virtual multipliers, which assure the economic viability of efficient DMUs, and/or satisfy other important considerations.

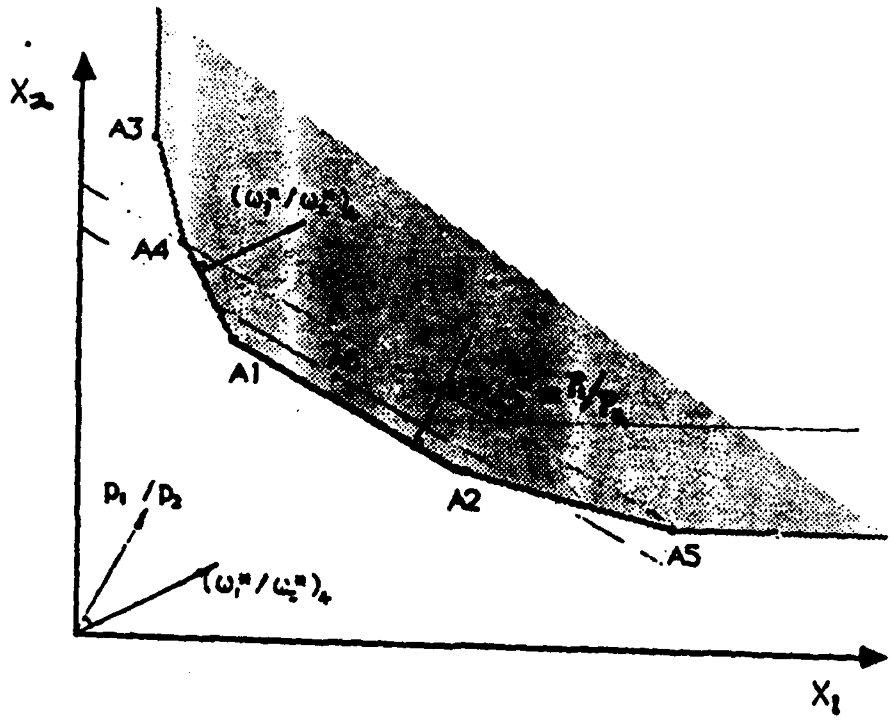

Fig. 1

I. THE CONE-RATIO DEA MODEL

In order to generalize to infinitely many DMUs and more general conditions that may impose restrictions on the dual evaluators of outputs and inputs, the CCR model as in Charnes, Cooper, Rhodes (l978), was generalized by Charnes, Cooper, Wei and Huang (l986) to the "Cone-Ratio CCR Model,"(here presented only for the finite number of DMUs case). See CCS 559, Center for Cybernetic Studies, The University of Texas at Austin, January, l987.

(l)
$$
\begin{aligned}
V_p = \text{Max} \quad & \mu^T \overline{Y}_0 \\
\text{s.t.} \; -\omega^T \overline{X} + & \mu^T \overline{Y} \le 0 \\
\omega^T \overline{X}_0 \quad & = 1 \\
\omega \in V, \; & \mu \in U
\end{aligned}
$$

and its dual (in the DEA form),

(2)
$$
\begin{aligned}
V_D = \text{Min } \theta & \\
\text{s.t.} \quad -\overline{X}\lambda + \theta \overline{X}_0 & \in -V^* \\
\overline{Y}\lambda \qquad -\overline{Y}_0 & \in -U^* \\
\lambda \ge 0 &
\end{aligned}
$$

where $\overline{X}$ is the mxn input matrix for the n DMUs to be considered and $\overline{Y}$ is the sxn matrix of their outputs. $V \subseteq E_+^m$, $U \subseteq E_+^s$ are closed convex cones and V^* and U^* are the negative polar cones of U and V, respectively.

We use $\overline{X}_j$ and $\overline{Y}_j$ respectively to represent the input vector and output vector of the j^{th} DMU and assume that $\overline{X}_j \in \text{Int}(-V^*)$, $\overline{Y}_j \in \text{Int}(-U^*)$ for any j. $\text{Int}(-V^*) = \{v: v'v > 0, \text{ for all } v' \in V \text{ and } v' \neq 0\}$. $\text{Int}(-U^*) = \{u : u'u > 0, \text{ for all } u' \in U \text{ and } u' \neq 0\}$. $\text{Int}(V^*)$

and Int(U^*) are not empty since V and U are "acute" cones as defined via $V \subseteq E^m_+$, $U \subseteq E^s_+$. See Yu (l974).

Both (l) and (2) have optimal solutions. With suitable regularity conditions,[1] the optimal solutions are equal, $V_D = V_P = \mu^{*T}\overline{Y}_0 \leq \omega^{*T}\overline{X}_0 = 1$.

Definition l: DMUo is said to be efficient if there exists an optimal solution (μ^*, ω^*) of (l) such that

$$\mu^{*T}\overline{Y}_0 = 1$$

and

$$\mu^* \in \text{Int}\, U, \; \omega^* \in \text{Int}\, V$$

The cone-ratio CCR model thus extends the CCR model by employing closed convex cones U and V which need not be nonnegative orthants. If we set $V = E^m_+$, $U = E^s_+$, then the two models coincide. See Charnes, Cooper, Wei and Huang (l986) for further discussion.

Polyhedral Cones V and U and the Cone-Ratio CCR Model

As long as there are only a finite number of inputs, outputs and DMUs, it may suffice to employ only polyhedral cones V and U to achieve desirable variants of past DEA efficiency evaluations. Polyhedral convex cones V and U may be expressed as

$$V = \{A^T\alpha : \alpha \geq 0\}, \quad \alpha \in E^{\ell}_+ \;, A^T = (a^1, a^2, \ldots, a^{\ell}), \; a^i \in E^m_+ \,, i = 1, \ldots, \ell;$$

(3)

$$U = \{B^T\gamma : \gamma \geq 0\}, \quad \gamma \in E^k_+ \;, B^T = (b^1, b^2, \ldots, b^k), \quad b^r \in E^s_+ \,, r = 1, \ldots, k;$$

and $V^* = \{v: Av \leq 0\}$, and $U^* = \{u: Bu \leq 0\}$.

Construction of a polyhedral convex cone V may be illustrated by the following example. Suppose the DMUs have two inputs. In the CCR model, the ratio of their

[1] Discussion of the conditions required to eliminate the possibility of a duality gap can be found in Huang (l985).

marginal substitution rate is $0 < \omega^*_1 / \omega^*_2 < \infty$, where * means optimal. Now suppose market information sets the range of this ratio as $c_1 \le \omega_2^* / \omega_1^* \le c_2$, with $c_2 \ge c_1 > 0$. This can be rewritten as

$$
(4) \quad \begin{aligned} -\omega_2^* + c_2\,\omega_1^* &\ge 0 \\ \omega_2^* - c_1\,\omega_1^* &\ge 0\,. \end{aligned}
$$

Thereby, $\omega^* \in V = \{\omega : C\omega \ge 0\}$ where

$$
C = \begin{bmatrix} c_2 & -1 \\ -c_1 & 1 \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}.
$$

V may also be defined equivalently as $\omega^* \in V = \{A^T\alpha : \alpha \ge 0\}$ then $-V^* = \{v: Av \ge 0\}$. Where

$$
A = \begin{bmatrix} 1 & c_1 \\ 1 & c_2 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.
$$

As will later be seen, polyhedral cones can tighten efficiency criteria in DEA tests. Before giving examples, we further expose some theoretical underpinnings as follows.

With U and V as polyhedral convex cones represented in the above form, then, using (3), problems (l) and (2) can be transformed into,

$$
(5) \quad \begin{aligned} & V_{p_1} = \text{Max } \gamma^T(B\overline{Y}_0) \\ & \text{s.t. } -\alpha^T(A\overline{X}) + \gamma^T(B\overline{Y}) \le 0 \\ & \qquad \alpha^T(A\overline{X}_0) = 1 \\ & \alpha \ge 0,\ \gamma \ge 0 \quad \alpha \in E^{\ell} \text{ and } \gamma \in E^k \end{aligned}
$$

$$\begin{aligned}
(6) \qquad & V_{D_1} = \text{Min } \theta \\
& \text{s.t. } -(A\overline{X})\lambda + \theta(A\overline{X}_0) \geq 0 \\
& \qquad (B\overline{Y})\lambda - (B\overline{Y}_0) \geq 0 \\
& \qquad \lambda \geq 0
\end{aligned}$$

Letting $\overline{X}' = A\overline{X}$, $\overline{Y}' = B\overline{Y}$, the cone-ratio CCR model then coincides with a CCR model evaluating the same DMUs but with the transformed data $\overline{X}'$ and $\overline{Y}'$. Note that $\overline{X}'$ and $\overline{Y}'$ are strictly positive, since $a^{iT} \in V$, $b^{rT} \in U$, and $\overline{X}_j \in \text{Int}(-V^*)$, $\overline{Y}_j \in \text{Int}(-U^*)$, $j = 1, \ldots, n$; $i = 1, \ldots, \ell$; $r = 1, \ldots, k$.

The following theorem establishes the existence of efficient DMUs for the cone-ratio DEA model.

__Theorem 1__ There exists at least one efficient DMU with the cone-ratio CCR model provided that U and V are polyhedral cones.

__Proof:__ Since U and V are polyhedral cones, after the transformation shown in problem (5), $A\overline{X}_j > 0$ and $B\overline{Y}_j > 0$, $j = 1, \ldots, n$. There exists an optimal solution (γ^*, α^*) in (5) for any one of the n DMUs, denoted by DMU_0, such that

$V_{P_1} = \gamma^{*T}(B\overline{Y}_0) = 1$ and $\gamma^* > 0$ and $\alpha^* > 0$

Let $\omega^* = A^T\alpha^*$, $\mu^* = B^T\gamma^*$. Then $\mu^{*T}\overline{Y}_0 = \gamma^{*T}(B\overline{Y}_0) = 1$, and $\omega^{*T}\overline{X} - \mu^{*T}\overline{Y} = \alpha^{*T}(A\overline{X}) - \gamma^{*T}(B\overline{Y}) \geq 0$. Further, $\omega^* = A^T\alpha^* \in \text{Int}(V)$, and $\mu^* = B^T\gamma^* \in \text{Int}(U)$, since $\{A^T\alpha: \alpha > 0\} \in \text{Int}(V)$, and $\{B^T\gamma: \gamma > 0\} \in \text{Int}(U)$.

Q.E.D.

Since problem (6) and problem (2) are equivalent, an optimal solution (λ^*, θ^*) to (6) is also an optimal solution to problem (2). Furthermore, since $U \subseteq E^s_+$, $V \subseteq E^m_+$, then $E^s_+ \subseteq -U^*$ and $E^m_+ \subseteq -V^*$. Although the conditions for optimal solutions of problem (6) are more restrictive than those of the corresponding CCR model, if DMU_o is efficient according to problem (6) (that is, $V_D = 1$), it must be efficient for the corresponding CCR model.

Now let $T = \{(X, -Y) : (X, -Y) \in (\overline{X}\lambda, -\overline{Y}\lambda) + (-V^*, -U^*), \lambda \geq 0\}$ be the production possibility set. Then

Definition 2: $(\overline{X}o, -\overline{Y}_o) \in T$ is said to be a nondominated point of T associated with $V^* \times U^*$, if there exists no $(X,-Y) \in T$ such that

$$(X,-Y) \in (\overline{X}_o, -\overline{Y}_o) + (V^*, U^*),\ (X, -Y) \neq (\overline{X}_o, -\overline{Y}_o)$$

Given this definition, the following theorem is proved in Charnes, Cooper, Wei and Huang (1986):

Theorem 2.

Let $(\overline{X}_o, -\overline{Y}_o)$ be a nondominated point of T associated with $V^* \times U^*$. Then DMU_o is efficient.

We shall show that an efficient DMU rated by the CCR model will not be efficient as rated by the cone-ratio CCR model if its facet normal is not contained in the constraint cone which is employed. Consequently, by suitable choice of constraint cones we can reclassify efficient DMUs and compare results from the two models for further insight when desired.

Testing each of the DMUs with the CCR model determines both points on the efficient (empirical) production frontier and, for each facet, the convex combinations of associated efficient points, as well as an optimal dual problem solution which is the normal vector to a "bounding" supporting hyperplane containing the facet. We call these dual solutions the "facet normals".

Assume that we have M facets from efficiency evaluations using the CCR model. The normal of each facet is $a^i = (\mu^*, \omega^*)^T \in E_+^{m+s}$, $i = 1, \ldots, M$. The corresponding halfspaces are $A_i = \{Z: a^{iT}Z \geq b_i\}$., the corresponding bounding hyperplanes are $B_i = \{Z: a^{iT}Z = b_i\}$, $i = 1, \ldots, M$. Z is any of the observed vectors of inputs and negative outputs $(x,-y)^T$ which is contained in at least one A_i.

Let

$$A = \bigcap_{i=1}^{M} A_i, \quad A^j = \bigcap_{\substack{i=j \\ i \neq j}}^{M} A_i .$$

Suppose we use a subset of the facet normals as spanning vectors for a constraint cone W, say (by renumbering if necessary)

$$W = \left\{ \sum_{i=1}^{k} \lambda_i a^i ; \lambda_i \geq 0, i = 1, \ldots, k \right\}, 1 \leq k \leq M.$$

Note that such a W is an acute cone, and thereby $Int(W^*)$ is not empty.

Lemma I below (now to be proved), shows that if a facet normal is not in W, then the associated efficient DMUs are no longer efficient under the cone-ratio model associated with W. In other words, these DMUs are dominated in the negative polar W^*.

Lemma I

If a^j is not in W, and $Z_0 \in B_j \cap Int(A^j)$, then Z_0 is not a nondominated point of A associated with W^*. I.e., there exists $Z \in A$, such that $Z \in Z_0 + W^* / \{0\}$.

Proof: Suppose, on the contrary, that there is no $Z \in A$ such that $Z \in Z_0 + W^* / \{0\}$. Let $S = \{s: s \in W^* / \{0\} - Z + Z_0$, for some $Z \in A\}$. It is easy to show that S is a convex set and 0 is not in S. By the separating hyperplane theorem for convex sets, there exists nonzero $p \in E^{m+s}$ such that $p^T s \leq 0$ for all $s \in S$.

For any $Z \in A$, $\lambda > 0$ and $w \in W^* / \{0\}$, let $S_{Z,\lambda,w} = -Z + Z_0 + \lambda w$. Then $p^T Z_0 + \lambda p^T w \leq p^T Z$ for all $Z \in A$, $\lambda > 0$, and $w \in W^* / \{0\}$. Hence,

(a) $p^T Z_0 \leq p^T Z$ for all $Z \in A$

(b) $p^T w \leq 0$ for all $w \in W^* / \{0\}$

From (b), since W is an acute cone, $p \in (W^* / \{0\})^* = W$.

Now consider the system

$$\text{(c)} \quad \begin{cases} p^T Z < 0 \\ a^{jT} Z = 0 \end{cases} .$$

There must exist a solution $\bar{Z}$ to (c). Otherwise, for all Z satisfying $a^{jT} Z = 0$, we would have $p^T Z = 0$. In that event, there must exist a scalar h such that $a^{jT} = hp$ with $h > 0$ since $a^j \geq 0$, and $p \geq 0$. This leads to $a^j \in W$, which contradicts our assumption.

Now let $\bar{Z}$ be a solution to (c) and consider the point $(Z_0 - \beta\bar{Z})$. $Z_0 \in \text{Int}(A^j)$ and $a^{iT} Z_0 > b_i$ for $i \neq j$. There exists $\alpha < 0$ such that for $i \neq j$, $a^{iT} (Z_0 - \beta\bar{Z}) = a^{iT} Z_0 - \beta a^{iT}\bar{Z} \geq b_i$ for all $\beta \in [\alpha, 0)$ and $a^{jT} (Z_0 - \beta\bar{Z}) = a^{jT} Z_0 - \beta a^{jT} \bar{Z} = a^{jT} Z_0 = b_j$. That means $(Z_0 - \beta\bar{Z}) \in A$. But, $p^T(Z_0 - \beta\bar{Z}) = p^T Z_0 - p^T \bar{Z} < p^T Z_0$ for all $\beta \in [\alpha, 0)$, that contradicts (a). Hence, there exists $Z \in A$ such that $Z \in Z_0 + W^* / \{0\}$.

Q.E.D.

From $a^{jT} (Z_0 - \beta\bar{Z}) = b_j$ we know Z_0 is in fact dominated by another DMU that is an extreme point located on the same facet. Note that Z_0 is not an extreme point of A^j since $Z_0 \in \text{Int}(A^j)$. Hence, from Lemma I, we immediately conclude

Theorem 3

A DMU which is evaluated as efficient by the CCR model is inefficient with the cone-ratio CCR model if its facet normal is not in the constraint cone employed.

We proceed next to

Theorem 4

An efficient DMU which is evaluated by the CCR model is still efficient under the cone-ratio CCR model if (μ^*, ω^*), its optimal dual solution to the CCR model is in the constraint cone (U,V) of the cone-ratio CCR.

Proof: Consider problem (I) for the cone-ratio model and the same problem for the CCR model which replaces the conditions $\omega \in V$, $\mu \in U$ in (I) with $\omega, \mu > 0$. Let (μ^*, ω^*) be an optimal solution to the latter problem. (μ^*, ω^*) is a facet normal that is in the constraint cone, i.e., $\mu^* \in \mathrm{Int}(U)$, $\omega^* \in \mathrm{Int}(V)$. Hence, it is a feasible solution to (I). Since the optimal functional value of (I) ≤ 1 and $\mu^* \overline{Y}_0 = 1$ in the CCR model, (μ^*,ω^*) must be an optimal solution to problem (I)

Q.E.D.

These theoretical conclusions are of practical importance. They make it possible for us to employ expert knowledge for evaluation in a DEA analysis and to do so without unduly straining that knowledge. For example, we can use the desirable input-output structure of economically viable DMUs as spanning vectors for the constrained cone and thereby evaluate economically efficient DMUs. The input-output structure of a DMU can be represented by its optimal dual evaluator vector. We shall illustrate by using this approach in the evaluation of the managerial performance of commercial banks in Section 3.

2. SELECTION OF CONES FOR VARIOUS PURPOSES

We now employ the polyhedral cones U and V to tighten the criteria for efficiency evaluations of DMUs. These cones may be classified further into (a) those which emphasize individual inputs and/or outputs; and (b) those which favor individual DMUs.

Classification (a). Cones Emphasizing Inputs and Outputs

Example I

We are to evaluate 4 DMUs which use two inputs to produce one output. The observed data are:

	DMU	1	2	3	4
$\overline{X}$	$\overline{x}_1$	1	4	1.5	4
	$\overline{x}_2$	5	1	1.5	2
$\overline{Y}$	$\overline{y}$	1	1	1	1

The CCR model will evidently identify DMUl, DMU2 and DMU3 as efficient. Let us examine their efficiency again with a polyhedral cone-ratio model. For simplicity, we constrain only cone V and set $U = E^1_+$.

Let $V = \{A^T\alpha: \alpha \geq 0\}$, then $-V^* = \{\omega: A\omega \geq 0\}$.

$$A^T = \begin{bmatrix} 1 & a^2 \\ a^1 & 1 \end{bmatrix}$$

so we have $-V^* = \{ \omega: \omega_1 + a^1 w_2 \geq 0, a^2 \omega_1 + w_2 \geq 0\}$. See Figure 2.

This is equivalent to using the CCR model to determine efficiency with the transformed inputs $\overline{X}' = A\,\overline{X}$ and the original output $\overline{Y}$ as represented in the following arrangement.

	DMU	1	2	3	4
$\overline{X}'$	$\overline{x}_1'$	$1+5a^1$	$4+a^1$	$1.5+1.5a^1$	$4+2a^1$
	$\overline{x}_2'$	a^2+5	$4a^2+1$	$1.5a^2+1.5$	$4a^2+2$
$\overline{Y}$	$\overline{y}$	1	1	1	1

If a^1 is sufficiently small and a^2 sufficiently large, the transformed data are equivalent to

	DMU	1	2	3	4
$\overline{X}'$	$\overline{x}_1'$	1	4	1.5	4
	$\overline{x}_2'$	a^2	$4a^2$	$1.5a^2$	$4a^2$
$\overline{Y}$	$\overline{y}$	1	1	1	1

since a^1 is dominated by the observed value of $\overline{x}_1'$ and a^2 dominates the observed value of $\overline{x}_2'$. Only DMU_1 which originally used the least $\overline{x}_1'$ can survive the efficiency test under the constrained cone.

Conversely, if a^2 is sufficiently small and a^1 sufficiently large, only DMU2 which originally used the least $\overline{x}_2'$ will remain efficient. For the same reason, the transformed data are equivalent to

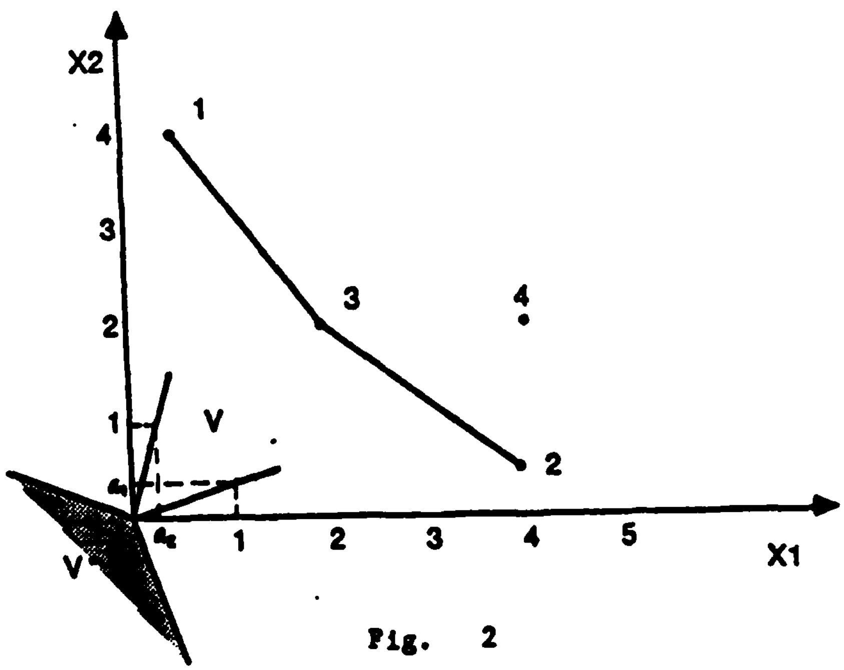

Fig. 2

	DMU	1	2	3	4
$\overline{X}'$	$\overline{x}_1'$	$5a^1$	a^1	$1.5a^1$	$2a^1$
	$\overline{x}_2'$	5	1	1.5	2
$\overline{Y}$	$\overline{y}$	1	1	1	1

In the first case, V is heavily tilted toward input $\overline{x}_1'$. This shows that more emphasis is now put upon $\overline{x}_1'$. As a result, conserving $\overline{x}_1'$ becomes of key concern. It is not strange that only DMUl, which consumed the least $\overline{x}_1'$, can survive this condition seriously favoring $\overline{x}_1'$. On the other hand, in the second case, emphasis is directed to $\overline{x}_2'$ and it makes conservation of $\overline{x}_2'$ much more desirable. Hence only DMU2 remains efficient. In the graph, cone V tilted toward axis $\overline{x}_2'$ in case two. See Figure 3a and 4a.

A convenient way to interpret the implication of these cones is to link them to the nondominated solution in the multi-objective programming problem. DMU2 and DMU3 are dominated in the polar cone $-V^*$ by DMUl in case one; DMUl and DMU3 are dominated by DMU2 in $-V^*$ in case two. (Fig. 3b, 4b).

We see from the above examples that a constraint cone tilted toward any objective (input and/or output) emphasizes that objective. This provides us with the possibility of taking account of different concerns for objectives which may not be explicitly rendered in the observed quantities themselves.

Classification (b). Cones Favoring DMUs

First, let us look at a special case that excludes "weak efficient" DMUs, i.e., a case that ensures strict positivity of (μ^*, ω^*) in problem (I). Then, DMU_0 is efficient if $\mu^{*T}Y_0 = 1$. We need to construct a constraint cone to exclude the hyperplanes $(0,a_2,\ldots,a_n)$, $(a_1,0,\ldots,a_n)$, $\ldots$, $(a_l,a_2,\ldots,0)$, but to include $(0,0,\ldots,0)$. So we may set

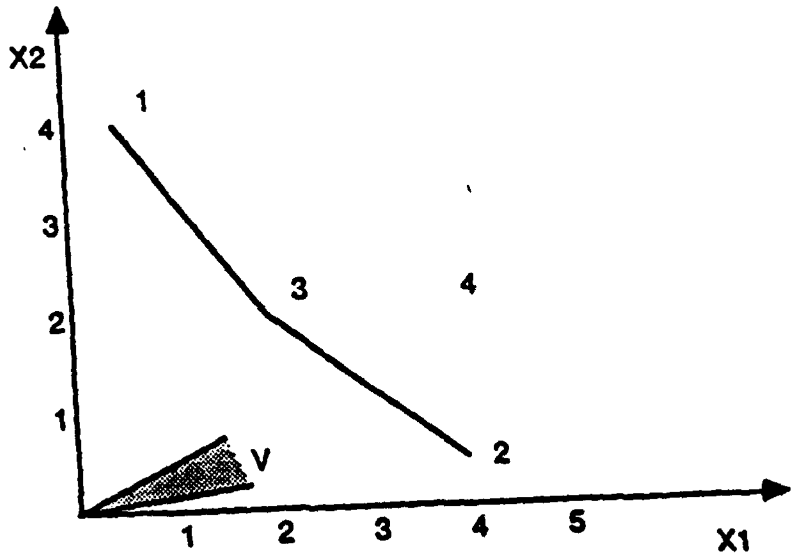

Fig. 3a

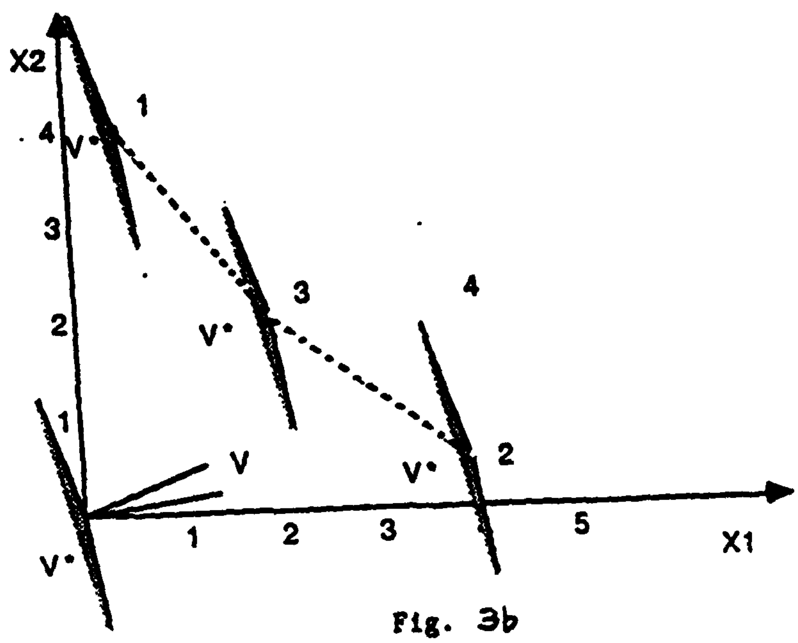

Fig. 3b

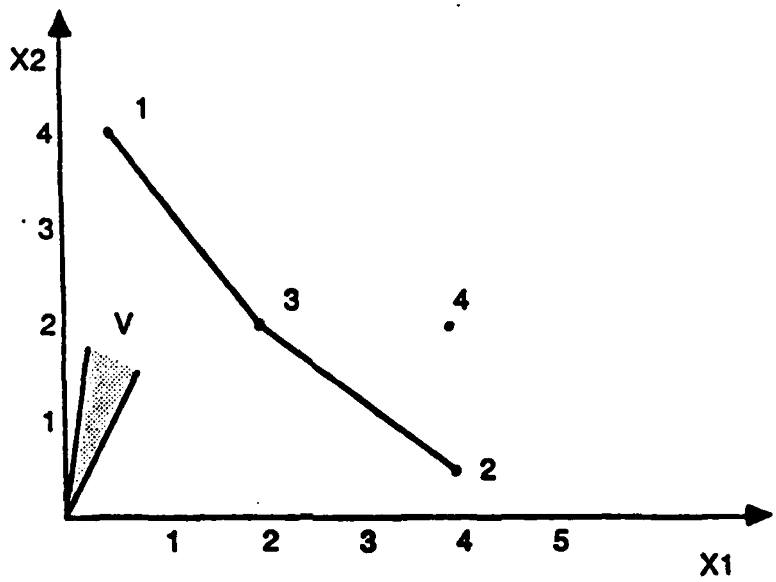

Fig. 4a

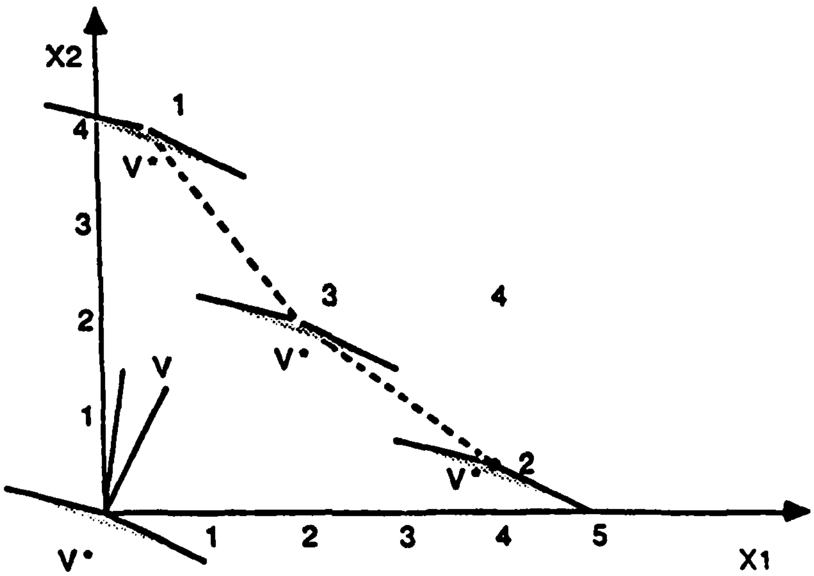

Fig. 4b

$$A_{mxm} = \begin{bmatrix} 1 & 1 & . & . & . & . & 1 & \varepsilon \\ 1 & 1 & . & . & . & . & \varepsilon & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon & 1 & . & . & . & . & 1 & 1 \end{bmatrix}$$

where ε is an arbitrarily small positive number. B_{sxs} is of the same matrix form for output transformation.

Example 2 is designed to illustrate how to detect the weak efficient DMUs using the above approach.

Example 2

We are to evaluate the performance of 17 DMUs. Each DMU uses 2 inputs to produce 1 output. The observed data are

DMU	$\bar{y}$	$\bar{x}_1$	$\bar{x}_2$		$\bar{y}$	$\bar{x}_1$	$\bar{x}_2$
DMU1	2	10	10	DMU10	2	4	30
DMU2	2	20	5	DMU11	2	6	15
DMU3	2	30	4	DMU12	2	6	15
DMU4	2	27	9	DMU13	2	7	13
DMU5	2	14	8	DMU14	2	40	5
DMU6	2	5	20	DMU15	2	20.5	4.9
DMU7	2	4	20	DMU16	2	4.1	19.5
DMU8	2	12	18	DMU17	2	5	15
DMU9	2	8	12				

Since the DMUs are all at the same output level, we can draw an isoquant curve in the input space as shown in Fig. 5. Consider DMU3 and DMU10. They are scale but not technically efficient, i.e., $\theta^* = 1$ but the slacks of inputs are not all zero. Specifically, the slack of input 1 is 5 for DMU3 and the slack of input 2 is 10 for DMU10. While they may be termed "weak efficient", they are not really efficient at all. But DMU3 and DMU10 seem to be fully efficient. (The θ^* of both are listed as 1.0000.) If the slack is not large, the product of it and a very small real value standing for ε (e.g., a choice of 10^{-6} in the computer code) can produce round-off effects. However, we can use the polyhedral constraint cone described above to uncover the true inefficiency of these DMUs as follows. Take

$$\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & .01 \\ 0 & .01 & 1 \end{bmatrix}$$

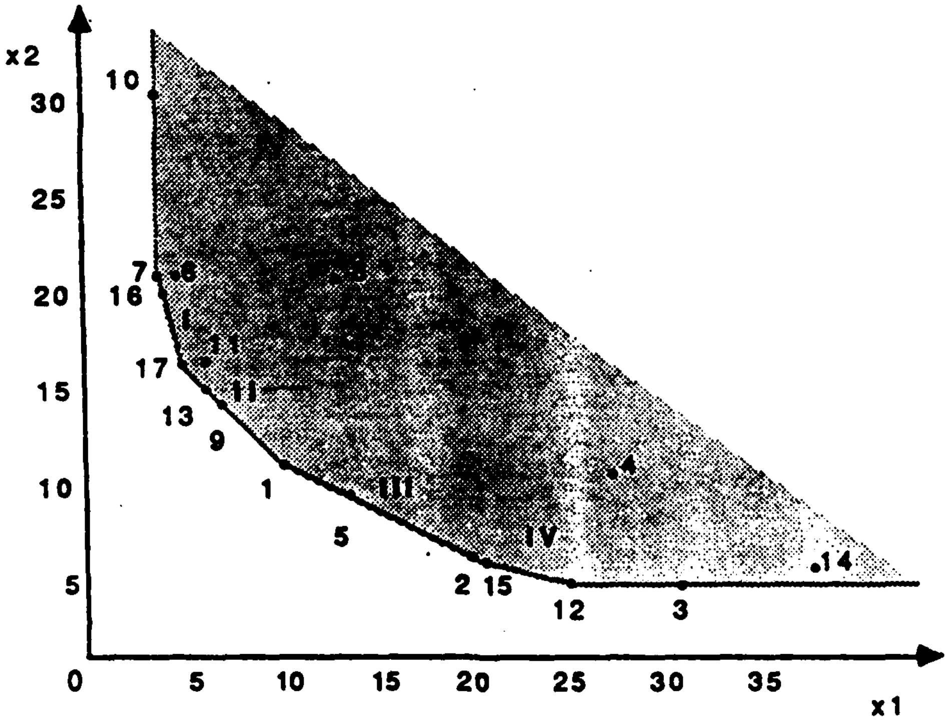

Fig. 5

With the transformed data (BY,AX), we get θ^* of DMU3 and DMUI0 as .9884 and .9767, respectively. Their inefficiencies are thus disclosed as in Fig. 6.

The next example illustrates an application of the polyhedral cone to identify the economically efficient DMUs.

Example 3

We use the same data as in Example 2. From the optimal solutions to the CCR model, we obtain four facet normals for the four facets. Now assume that market information indicates that the price ratio of inputs $\overline{x}_1$ and $\overline{x}_2$ are in the range k_l to k_2, and DMU managers want to adjust their input consumption accordingly. If $k_l = l/5$, and $k_2 = l$, only the efficient DMUs whose ω^*_2 / ω^*_1 are in the range (l/5, l) are economically efficient.

From Figure 7, we see that we can use the ω^* of facet I and facet II to establish the cone. Thus we take

$$\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & .125 & .025 \\ 0 & .05 & .05 \end{bmatrix}$$

Evaluating with the transformed data $(B\overline{Y}, A\overline{X})$, we obtain the new efficiency scores

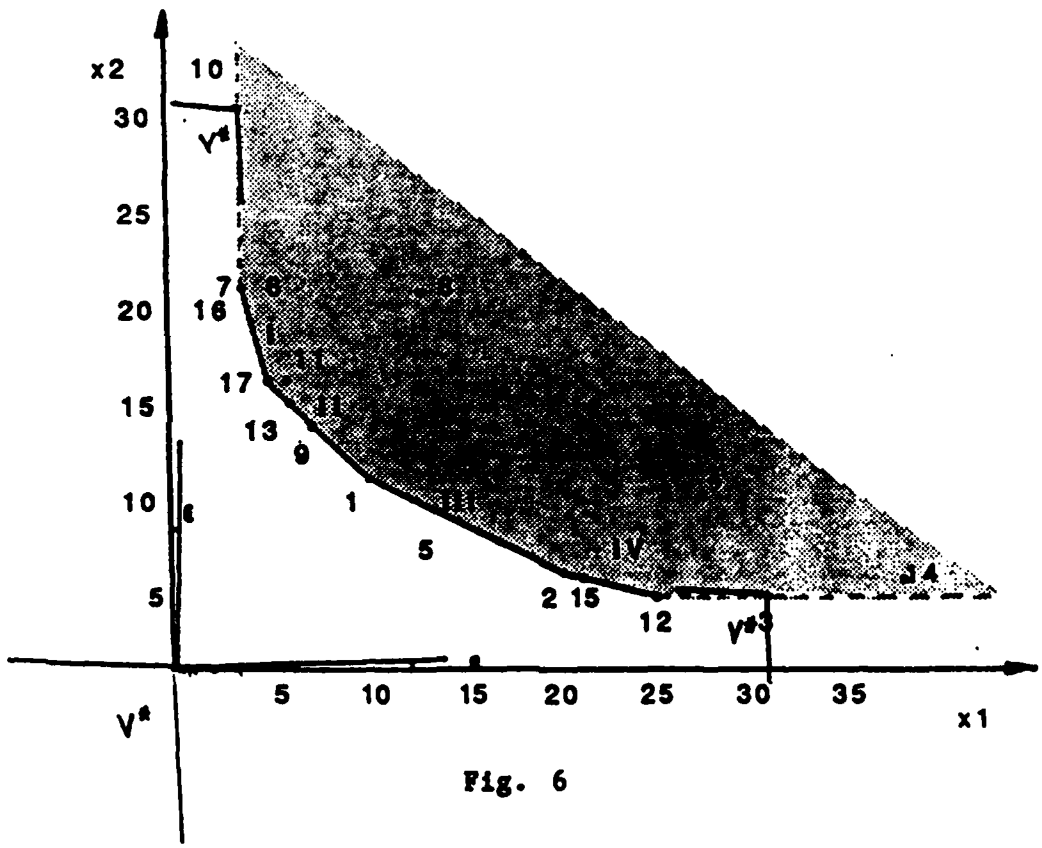

Fig. 6

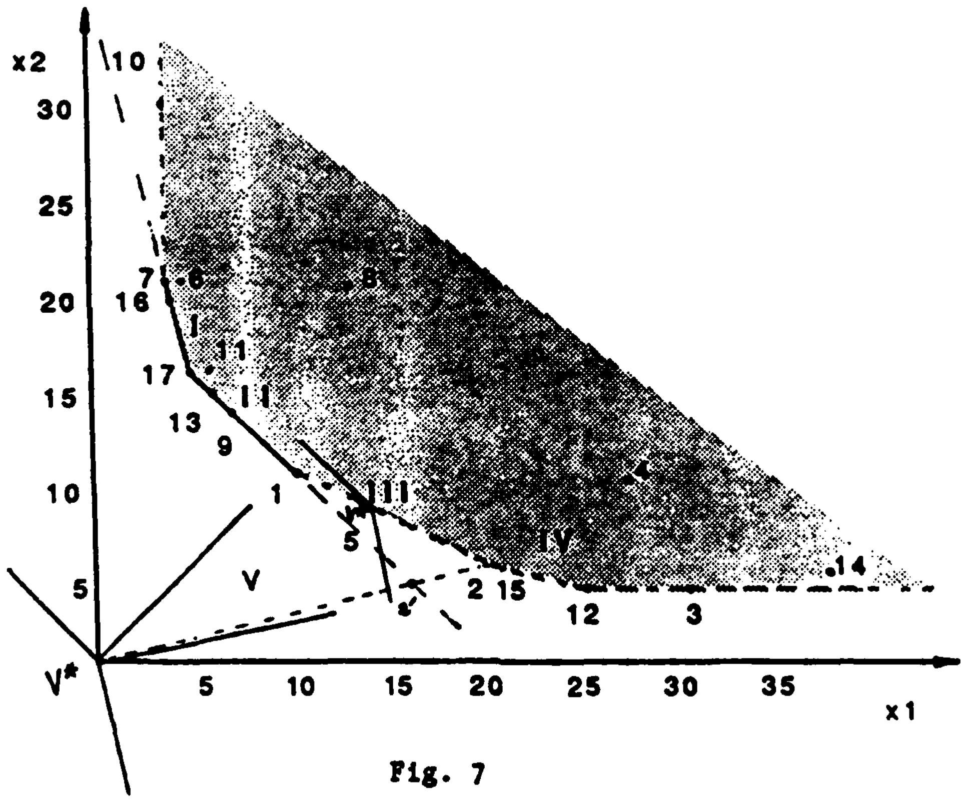
x2
30
25
20
15
10
5
V*
5
10
15
20
25
30
35
x1
10
7
6
8
16
I
11
17
II
13
9
1
III
5
V
4
IV
14
2
15
12
3

Fig. 7

DMU	SCORE	DMU	SCORE
DMUI	1.0000	DMU10	.3333
DMU2	.8000	DMU11	.9524
DMU3	.1923	DMU12	.6897
DMU4	.5556	DMU13	I.0000
DMU5	.909I	DMU14	.4444
DMU6	.8889	DMU15	.7874
DMU7	I.0000	DMU16	I.0000
DMU8	.6667	DMU17	I.0000
DMU9	I.0000		

For a two input case, as in this example, we may illustrate the constraint cone with the nondominated solutions associated with the negative polar of the constraint cone. Here, in Figure 7, we see that DMU2, DMU5, and DMUI5 are no longer efficient, since they are located on facets whose normal directions $(\omega^*_1, \omega^*_2)^T$ fall outside the range (I/5, I). But DMUI, which is originally on facet III, is still efficient. As a matter of fact, its ω^* could be any value between the normals of facet II and facet III. With the constraint condition, DMUI takes the normal of facet II as its new facet normal. To see this, note that DMUI is an extreme point. Recall that from Lemma I contrarily, $Z_0 = (\bar{x}_1, -\bar{y}_1)^T \in B_I$ but it is not in $Int(A^1)$.

Take DMU2 for a further example. Under the constraint cone, its efficient projection is to facet II where DMUI is located. Using the data for DMU_I, the bounding hyperplane is $(\bar{x}_1 - I0) = -(\bar{x}_2 - 10)$ while from the data for DMU_2, $4\bar{x}_1 = \bar{x}_2$. After solving, we therfore have $(\bar{x}_1, \bar{x}_2)^* = (4, I6)$, which is the efficient input usage for DMU2 under the constraint cone and the point on the frontier to which DMU2 projects. DMU2's new efficiency score is

$$0.8000 = \left[\frac{(4^2 + 16^2)^{1/2}}{(5^2 + 20^2)^{1/2}}\right]$$

From Figure 7, again, we can also easily see that the economically efficient input usage for DMU14 is (20/9, 160/9). It is the economically efficient point to which

DMU14 projects. The corresponding efficient score (.444) is the ratios of the distances of the efficient point to the observed point.

We have thus illustrated how a constraint cone can be selected to favor desired patterns of input usage and output production in efficiency evaluation. Further, as shown in Charnes,Cooper, Wei and Huang (l986) as well as in Sun (l987), these cone ratio approaches can be adapted for use with other models, such as the "additive" model, which embody the DEA concepts and methods of computation and analyses.

3. APPLICATIONS TO COMMERCIAL BANKS

We turn next to a realistic application to large commercial banks. As reported in Sun (l987), the data involved were drawn from the call reports (l980-l985) to the FDIC[1] for 48 U.S. commercial banks drawn from the top 300 banks headquartered in America which are also members of FDIC.

Using expert advice from a banking specialist the following outputs and inputs were used in this study:

Outputs:	Inputs:
1. Total Operating Income	1. Total Operating Expense
2. Total Interest Income	2. Total Non-Interest Expense
3. Total Non-Interest Income	3. Provision for Loan Losses
4. Total Net Loans	4. Actual Loan Losses

To be noted is that the provision for loan losses and actual loan losses treated as inputs are indicators of risks in banking operations. Total net loans is a measure of the size of services that a bank produces while the other inputs and outputs are mainly profit related measures.

[1]The supplemental data and expert opinions used are described in Sun (l987)

The results obtained from the CCR ratio model as applied to the data for these inputs and outputs were not satisfactory so recourse was made to a polyhedral cone-ratio DEA model with results that passed muster in subsequent reviews by experts with wide experience in banking.

Here we only provide a pair of examples to show what occurred and to show how the CCR model and its cone ratio extensions were used. For the first example, we use Citibank which, for 1983, showed the results listed under the column headed Value Observed. The column headed CCR model in this same Table shows the values for efficient performance as estimated by this model. The values exhibited under the column designated as cone-ratio CCR show the values which efficient performance would have exhibited as estimated with the cone-ratio CCR model.

As can be seen, the values in the latter two columns differ. The CCR model rated Citibank performance as efficient but the Cone-Ratio CCR model did not.[1] The value of $\theta^* = 0.9693$ obtained from the latter model applied to all of the observed input values produces the values shown for these same inputs in the last column with the result that these inputs are all reduced by about 3%.

Turning to the ouput values, we obtain the adjustments needed for efficiency attainment by means of the formula

$$\overline{Y}_0^* = \sum_{j=1}^{n} \lambda_j^* \overline{Y}_j$$

where the $\overline{Y}_j$ are the vectors of observed values which correspond to the efficient DMUs used in the evaluation of DMU_0 and the λ_j^* are the optimal solution values. $\overline{Y}_0^*$ is the value corresponding to the point on the efficient facet from which the outputs observed in Y_0 are evaluated.

[1]The cone-ratio additive model also rated Citibank as 100% efficient in 1983.

In the case of Citibank's 1983 performance, the banks appearing in the optimal basis--and thus the banks used in evaluating the efficiency of Citibank's performance--are the Republic National Bank of New York and Texas Commerce Bank, with λ_j^* values of 2.85 and 12.22, respectively. Applying these values to the 1983 data for these two banks produced the results for the output values shown in the upper part of the last column in Table 3. Comparison with the observed values for Citibank showed that this would have resulted in a decrease of total income by some 1%, a decrease in interest income of 6% and a decrease in non-interest income by 50% whereas net loans would have increased by 19%. To be noted, therefore, is the fact that the reduction of inputs (by some 3%) may then be accompanied by a decrease in some outputs and an increase in others.

TABLE 3

CITIBANK (1983)

	VALUE OBSERVED	VALUE IF EFFICIENT	
		CCR MODEL	CONE-RATIO CCR
OUTPUT			
Total income	13572000	13572000	13443860
Interest income	10615000	10615000	10020451
Noninterest income	553000	553000	271151
Net loans	69286000	69286000	82397984
INPUT			
Provisions	320000	320000	310176
Total expense	12171000	12171000	11797350
Noninterest Expense	3061000	3061000	2967027
Loan losses	263000	263000	254926

For another example, we turn to Continental Illinois for 1984 which is known to have been a disastrous year for this bank. The data for this case and the corresponding CCR model and cone-ratio CCR model estimated efficiency adjusted values are shown in Table 4 which has the same arrangement as Table 3.

In this case, the CCR model gave a value of $\theta^* = 0.919$ which was reduced to 0.2351 by the cone-ratio version of this model. Evidently a drastic reorientation of this bank's activities is signaled by the latter value, as was subsequently confirmed by the complete overhaul initiated with the FDIC bail-out attempt for Continental Illinois.

Turning from the inputs to the outputs for Continental Illinois in 1984, we observe that Wachovia National Bank and Trust Co. is the only DMU appearing in the basis from which Continental Illinois was evaluated. Thus applying the value of $\lambda_j^* = 4.74$ to the data for Wachovia in 1984, we obtain the new Y_0^* output values for Continental Illinois which are shown in the last column of Table 4. Associated with this nearly 77% reduction in its inputs, as shown in Table 4, Continental Illinois might also have increased its total income by 5% and its interest income by some 14% and 147%, respectively, while decreasing its total loans by nearly 4%

TABLE 4

CONTINENTAL ILLINOIS NB & TC (1984)

	VALUE OBSERVED	VALUE IF EFFICIENT	
		CCR MODEL	CONE-RATIO CCR
OUTPUT			
Total income	3998187	3998187	4209945
Interest income	3334291	3334291	3380729
Noninterest income	70064	96783	172986
Net loans	23693936	24791577	22922308
INPUT			
Provisions	1171878	143749	275509
Total Expense	3703887	3405380	870784
Noninterest expense	779890	717036	183352
Loan losses	1165487	96907	274006

Figure 8 provides a geometric portrayal which can illustrate what is happening in the above cases. As is evident from these examples, output adjustments to attain efficiency in the case of the cone-ratio model need not be limited to movement in the "northeast" direction, as is true for the CCR model. Thus, in the case shown in Figure 8, the output adjustment for DMU5 is restricted to projections on AA'. In the cone-ratio CCR model, however, the projections can be to BB'.

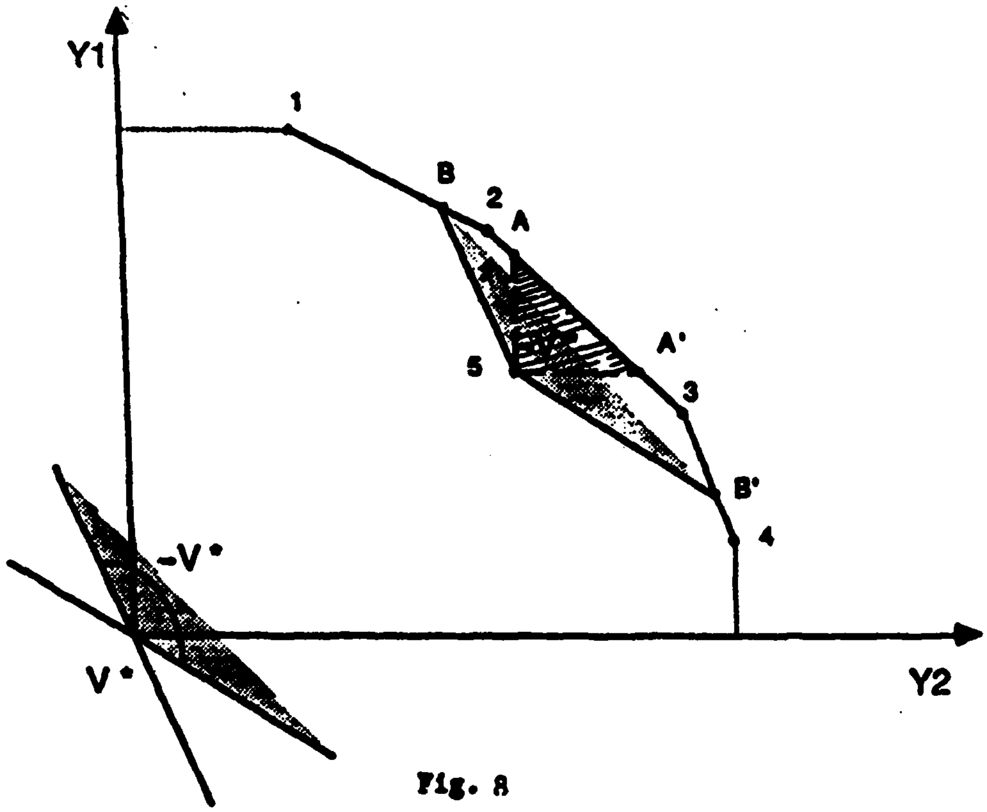

Fig. 8

SUMMARY AND CONCLUSIONS

This extract from a more extended study should help to show some of the differences that may be expected as opportunities vistas for research and use are opened by the cone-ratio extensions of DEA. Evidently a good deal of flexibility is added and ways are opened for the use of expert opinion without strain since a knowledge of only ranges of values with associated inequalities need be employed. Needless to say, these uses can also provide guidance and act as a control on such opinions.[1]

These cone ratio developments open other possibilities as well. For instance, the deficiencies exhibited by the ordinary CCR ratio model may reflect rather the fact that FDIC call report data are insufficient to provide all of the indicators needed to distinguish between efficient and inefficient performance.[2] Indeed, as shown in Charnes, Cooper, Golany, Halek, Schmitz and Thomas (1986), uses of DEA admit of extensions that include "goals" which might be specified for attainment as well as laws or regulations, risk factors and/or economic "climate". Finally, cone-ratio extensions can be applied to the elimination of activities and/or merger schemes along the lines of what was done in Bessent, Bessent, Charnes, Cooper and Thorogood (1983).

In any case, the basic ideas and principles have been set forth in the preceding discussion and the mathematical details have been set forth with full rigor in Charnes, Cooper, Wei and Huang (1986).

[1]See, e.g., the discussion in Thomas, 1986, of the way DEA was used to guide and evaluate the performance of the auditors of the Texas Public Utility Commission in their managerial audits.

[2]See the similar comments in Divine (1986) on the use of DEA for effecting bond-rating evaluations for electric utilities which are more comprehensive than the ratings provided in Standard and Poor's or other bond rating services.

REFERENCES

Bessent, A., W. Bessent, A. Charnes. W. W. Cooper and N. Thorogood (1983), "Evaluation of Educational Proposals by Means of DEA," Educational Administration Quarterly, 19, pp. 82-92.

Charnes, A. and W. W. Cooper (1985), "Preface to Topics in Data Envelopment Analysis," Annals of Operations Research, 2, pp. 59-94.

__________, __________, B. Golany, R. Halek, G. Klopp, E. Schmitz and D. Thomas (1986), "Data Analysis Approaches to Policy Evaluation and Management of Army Recruiting Activities I: Tradeoffs Between Joint Services and Army Advertising," Research Report 532, (Center for Cybernetic Studies, The University of Texas at Austin, 1986).

__________, __________, __________, L. Seiford and J. Stutz, (1985), "Pareto-Koopmans Optimality, Efficiency Analysis and Empirical Production Functions," Journal of Econometrics 30, (1985), pp. 91-107.

__________, __________, and E. Rhodes (1978), "Measuring the Efficiency of Decision Making Units," European Journal of Operations Research, Vol. 2, No. 6, (Nov.1978), pp. 429-444.

__________, __________, __________, (1981), "Data Envelopment Analysis as an Approach for Evaluating Program and Managerial Efficiency -- With an Illustrative Application to the Program Follow Through Experiment in U.S. Public School Education," Management Science 27, No. 6 (June, 1981) pp. 668-697.

__________, __________, J. Rousseau and J. Semple (1978), "Data Envelopment Analysis and Axiomatic Notions of Efficiency Reference Sets," CCS Research Report 558, (Center for Cybernetic Studies, The University of Texas at Austin, 1987).

__________, __________, Q. L. Wei and Z. M. Huang, (1986), "Cone Ratio Data Envelopment Analysis and Multi-Objective Programming," (The University of Texas at Austin, Center for Cybernetic Studies Report CCS 559, January 1987), International Journal of Systems Science (to appear).

Divine, J.D., (1986), "Efficiency Analysis and Management of Not-For-Profit and Governmentally Regulated Organizations," Ph.D. Thesis (Austin: The University of Texas at Austin).

Huang, Z.M. (1985), "The Second Order Conditions of Nondominated Solutions for Generalized Multiobjective Mathematical Programming," Journal of Systems Science and Mathematical Science, Vol. 5, No. 3, pp. 172-184.

Sun, D. B. (1987) "Evaluation of Managerial Performance of Large Commercial Banks by Data Envelopment Analysis," Unpublished Ph.D. Dissertation, (Austin: Graduate School of Business, The University of Texas at Austin).

Thomas, D. L. (1986), "Auditing the Efficiency of Regulated Companies: An Application of Data Envelopment Analysis to Electric Cooperatives," (Austin: The University of Texas at Austin, IC^2 Institute).

Yu, P.L., "Cone Convexity, Cone Extreme Point, and Nondominated Solutions in Decision Problems with Multi-Objectives," Journal of Optimization Theory and Application, Vol. 14, No. 3, 1974.